

|        |                                                    |
|--------|----------------------------------------------------|
| Title  | Junior College H2 Mathematics Material Compilation |
| Author | AprilDolphin                                       |
| Date   | 27/4/2025                                          |

| Page | Topic                                                               |
|------|---------------------------------------------------------------------|
| 2    | Graphs and Functions – Conic Sections                               |
| 5    | Inequalities – Involving Quadratic, Cubic and Rational Functions    |
| 11   | Inequalities – Involving Modulus Functions                          |
| 17   | Differentiation – Involving Inverse Trigonometric Function          |
| 19   | Differentiation – Implicit Differentiation                          |
| 23   | Differentiation – Parametric Differentiation                        |
| 26   | Integration – Integration by Parts                                  |
| 29   | Differential Equations – Basic Concepts and Separation of Variables |

|        |                                                                                                                                                                                |
|--------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Title  | Graphs and Functions – Conic Sections – Hyperbola and Ellipse                                                                                                                  |
| Author | AprilDolphin                                                                                                                                                                   |
| Date   | 21/12/2024                                                                                                                                                                     |
| Note   | This article assume you already understand the properties of circle and parabola as taught in secondary school Additional Mathematics and Elementary Mathematics respectively. |

In general, all conic sections can be represented using the equation below,

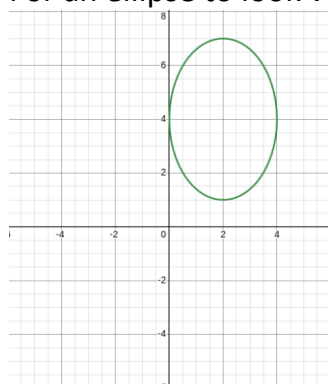
$$Ax^2 + Bx + Cy^2 + Dy + E = 0$$

Different types of conic sections appears when we begin changing the value of  $A, B, C, D$  and  $E$

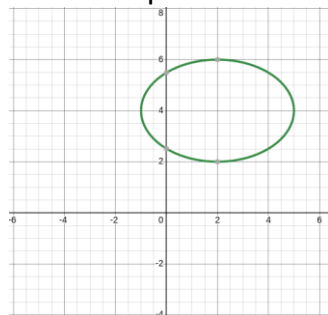
If the conic section is an Ellipse, it has the following equation required in general: **(Make sure the right-hand side is exactly 1 before proceeding!)**

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

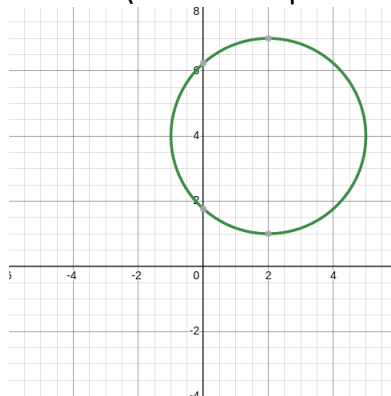
For an ellipse to look vertically stretched,  $a < b$



For an ellipse to look horizontally stretched,  $a > b$



For an ellipse to have an exactly circular shape,  $a = b$ , where  $a$  represents the radius of a circle. (Circle is a special case of an ellipse.)



Properties of an ellipse.

The point where the centre lies is  $C(h, k)$

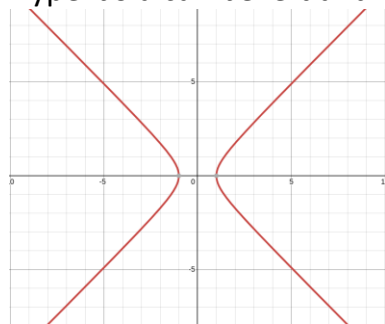
The coordinates of vertices from up, down, left and right can be represented as the following

|                                      |               |
|--------------------------------------|---------------|
| Upper Vertex (Top of Ellipse)        | $U(h, k + b)$ |
| Lower Vertex (Bottom of Ellipse)     | $D(h, k - b)$ |
| Left Vertex (Left side of Ellipse)   | $L(h - a, k)$ |
| Right Vertex (Right side of Ellipse) | $R(h + a, k)$ |

The lines of symmetry of an ellipse are represented by  $x = h, y = k$

If the conic section is a Hyperbola, it has the following equation in general: (Make sure the right-hand side is exactly 1 before proceeding!)

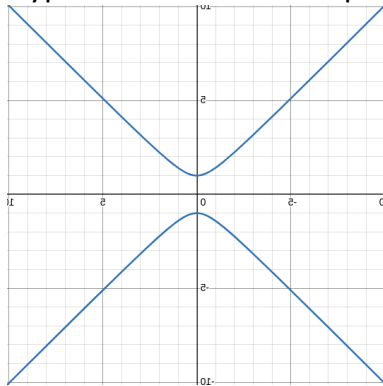
Hyperbola can be left and right opening as seen with the below diagram



Properties of left-and-right-opening hyperbola as follows:

Can be reduced to the following equation,  $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ , where RHS is exactly 1

Hyperbola can also be up-and-down-opening as seen with the below diagram



Properties of up-and-down-opening hyperbola as follows:

Can be reduced to the following equation,  $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ , where RHS is exactly 1

| Further Properties of Hyperbola |                                                              |                                                              |
|---------------------------------|--------------------------------------------------------------|--------------------------------------------------------------|
|                                 | Left-and-Right-Opening                                       | Up-and-Down-Opening                                          |
| Centre                          | $C(h, k)$                                                    | $C(h, k)$                                                    |
| Vertices                        | Left Vertex, $L(h - a, k)$<br>Right Vertex, $R(h + a, k)$    | Upper Vertex, $U(h, k + b)$<br>Bottom Vertex, $B(h, k - b)$  |
| Line of symmetry                | $x = h$<br>$y = k$                                           | $x = h$<br>$y = k$                                           |
| Asymptotes                      | $y = k + \frac{b}{a}(x - h)$<br>$y = k - \frac{b}{a}(x - h)$ | $y = k + \frac{b}{a}(x - h)$<br>$y = k - \frac{b}{a}(x - h)$ |

|        |                                                                |
|--------|----------------------------------------------------------------|
| Title  | Inequalities Involving Quadratic, Cubic and Rational Functions |
| Author | AprilDolphin                                                   |
| Date   | 3/2/2025                                                       |

### Flashback from Secondary School Elementary Mathematics

| Description of Rules                                                                                                                                                                                                                             | Demonstration                                                                                                                                                    |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Addition & Subtraction<br>Rule: The act of adding any numbers to both sides of any inequalities doesn't affect the sign.                                                                                                                         | $x > 8$<br>$x + 4 > 8 + 4$<br><br>$x \leq 9$<br>$x + 1 \leq 9 + 1$                                                                                               |
| Multiplication & Division of Positive Numbers Rule: The act of multiplying or dividing of <b>positive numbers</b> to both sides of any inequalities doesn't affect the sign.                                                                     | $5x + 8 \geq 9$<br>$\frac{5x}{5} \geq \frac{1}{5}$<br>$x \geq \frac{1}{5}$                                                                                       |
| Multiplication & Division of Negative Numbers Rule: The act of multiplying and dividing of <b>negative numbers</b> to both sides of any inequalities will flip the inequality sign (i.e. $\leq$ changes to $\geq$ and $\geq$ changes to $\leq$ ) | $2x > 6$<br>After dividing both sides by $-2$ , we get<br>$-x < -3$<br><br>$\frac{x}{3} \geq 5$<br>After multiplying both sides by $-3$ , we get<br>$-x \leq -5$ |

| Common Mistakes Made by Inexperienced Students when Solving Inequalities at H2 Level Math. |                                                                                                                                                                                                                                          |
|--------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Mistake Number 1.                                                                          | <p><b>Cross Multiplication</b></p> $\frac{x - 6}{2x + 3} > 1$ $x - 6 > 2x + 3$ <p>Issue: It is incorrect to cross multiply this way as it is currently not known if the sign of <math>2x + 3</math> is positive or negative.</p>         |
| Mistake Number 2.                                                                          | <p><b>Square Rooting Both Sides of Inequality</b></p> $x^2 \geq 9$ $x \geq \pm 3$ <p>Issue: It is not correct if you treat the inequality sign as if it were an equal sign for the purpose of square rooting both sides in this way.</p> |

Correct Steps to Deal with Inequalities at H2 Mathematics as follows (The steps are to be followed with discretion, as not all steps are required for all question types.)

1. Moving all the terms of inequalities from RHS to LHS, making sure the RHS of the inequality is 0 before proceeding to step 2.
2. Add up everything on LHS (If LHS now has multiple algebraic fractions, add them up into a state where the LHS shares a common denominator.)
3. Factorize the LHS [If LHS cannot be factored, apply quadratic formula to any quadratic portions of the inequality.]
4. If the quadratic portion has no real roots (such as  $x^2 + x + 1$ ), complete the square as such portion will be either always positive or always negative.
5. For each linear factor, find the critical value that satisfy the inequality
6. Mark all critical values on the number line,  $n$  critical values divide the number line into  $n + 1$  interval
7. Do a sign test on each interval to determine whether the function is positive or negative in that interval.
8. Complete the sign diagram
9. Determine the solution set, noting that any values that will cause the denominator to be zero must be excluded as division by zero is undefined.

Q1:

Solve the inequality  $x^3 \geq x(3x + 10)$ .

Step 1. Move terms from RHS to LHS of the inequality.

$$x^3 - x(3x + 10) \geq 0$$

Step 2. Add up everything on LHS of inequality

$$x^3 - 3x^2 - 10x \geq 0$$

Step 3. Take out  $x$  by factoring.

$$x(x^2 - 3x - 10) \geq 0$$

Step 4. Further factorize to get the following.

$$x(x + 2)(x - 5) \geq 0$$

Note: to get sign test result on the bottom right of this page, substitute  $x$  on LHS with sign test value.

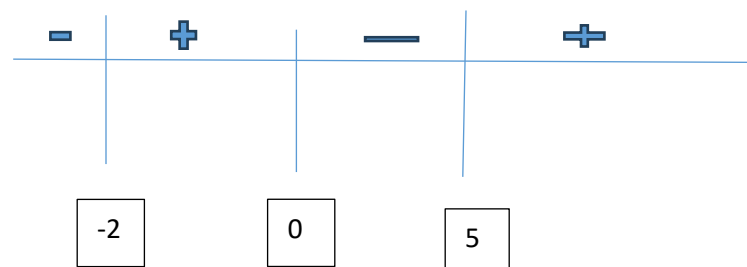
Example if your sign test interval is less than  $-2$ , find a value that is less than  $-2$ , in this case, I use  $-3$  and substitute  $x$  with  $-3$  to get

$$(-3)[(-3) + 2][(-3) - 5] = -24$$

Since the inequality in step 4 indicates the inequality has to be  $\geq 0$ , we look out for set of values that will produce positive or zero. Which get us the following as a result.

$$-2 \leq x \leq 0 \text{ OR } x \geq 5$$

| Sign Test | Result            |
|-----------|-------------------|
| -3        | -24<br>(Negative) |
| -1        | 6<br>(Positive)   |
| 1         | -12<br>Negative   |
| 6         | 48<br>(Positive)  |



Q2: Solve the inequality  $x^3 < 4x^2 - x$

Step 1. Move terms from RHS to LHS

$$x^3 - 4x^2 + x < 0$$

Step 2. Factorize the LHS of the inequality

$$x(x^2 - 4x + 1) < 0$$

Step 3. Use quadratic formula (which I will not further elaborate here) to deal with quadratic portion (Since it cannot be factorized).

$$x^2 - 4x + 1 = 0$$

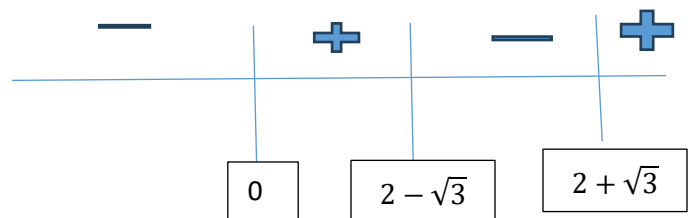
$$x = 2 \pm \sqrt{3}$$

Final Inequality:

$$[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})](x) < 0$$

$$x < 0 \text{ OR } 2 - \sqrt{3} < x < 2 + \sqrt{3}$$

| Sign Test | Result              |
|-----------|---------------------|
| -2        | -26<br>(Negative)   |
| 0.1       | 0.061<br>(Positive) |
| 3         | -6<br>(Negative)    |
| 4         | 4<br>(Positive)     |





Q3. Solve the inequality  $\frac{3x-5}{x-1} \geq \frac{2}{x}$

Step 1: Move all terms from RHS to LHS of inequality

$$\frac{3x-5}{x-1} - \frac{2}{x} \geq 0$$

Step 2: Add up everything on the LHS of inequality

$$\frac{x(3x-5)}{x(x-1)} - \frac{2(x-1)}{x(x-1)} \geq 0$$

$$\frac{3x^2 - 5x - 2x + 2}{x(x-1)} \geq 0$$

$$\frac{3x^2 - 7x + 2}{x(x-1)} \geq 0$$

Step 3. Factorize any quadratic portion of the LHS of inequality.

$$\frac{(3x-1)(x-2)}{x(x-1)} \geq 0$$

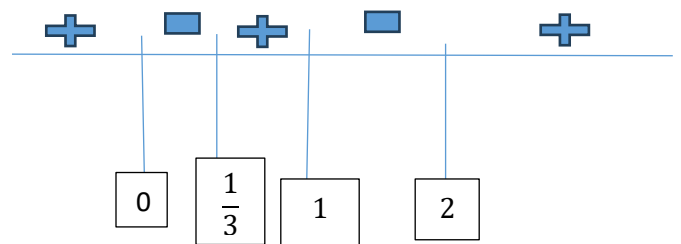
Step 4. Complete sign diagram and exclude any value that will cause the denominator to be 0.

$$x \neq 0 \text{ AND } x \neq 1$$

Therefore:

$$x < 0, \frac{1}{3} \leq x < 1 \text{ OR } x \geq 2$$

| Sign Test | Result            |
|-----------|-------------------|
| -1        | 6 (Positive)      |
| 0.25      | -2.333 (Negative) |
| 0.5       | 3 (Positive)      |
| 1.5       | -2.333 (Negative) |
| 3         | 1.333 (Positive)  |



Q4. Solve the following inequality

$$\frac{x+1}{x-1} \leq 4$$

Step 1. Move terms from RHS to LHS

$$\frac{x+1}{x-1} - 4 \leq 0$$

Step 2. Add up all LHS terms together in the following way

$$\frac{x+1}{x-1} - \frac{4(x-1)}{x-1} \leq 0$$

$$\frac{x+1-4x+4}{x-1} \leq 0$$

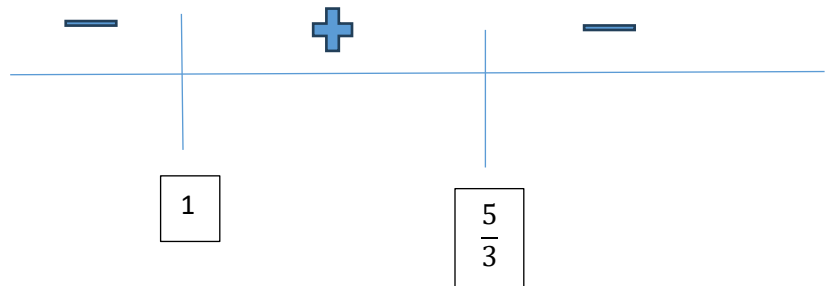
$$\frac{-3x+5}{x-1} \leq 0$$

Step 3 Complete sign diagram and exclude values that will cause the denominator to be 0.

$$x \neq 1$$

Therefore

$$x < 1 \text{ OR } x \geq \frac{5}{3}$$



| Sign Test | Result        |
|-----------|---------------|
| 0         | -5 (Negative) |
| 1.5       | 1 (Positive)  |
| 2         | -1 (Negative) |

|        |                                         |
|--------|-----------------------------------------|
| Title  | Inequalities Involving Modulus Function |
| Author | AprilDolphin                            |
| Date   | 14/2/2025                               |

### Basic Concepts of Modulus Functions

The absolute symbol of modulus functions will cause any negative values within the symbol to become positive and any positive values within the absolute symbol remain positive as illustrated in the following examples:

$$|\pi| = \pi$$

$$|-6| = 6$$

$$|-e^3| = e^3$$

$$|15| = 15$$

### Basic Properties of Equation Involving Modulus Functions

If  $|f(x)| = k$ , it will also imply  $f(x) = \pm k$

### Basic Properties of Inequalities Involving Modulus Functions

If  $|f(x)| < k$ , it will also imply  $-k < f(x) < k$

If  $|f(x)| > k$ , it will also imply  $f(x) < -k$  OR  $f(x) > k$

### Basic Properties of Operations Involving Modulus Functions

Given  $a, b$  are real numbers, the following properties shall apply

$$|ab| = |a||b|$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

$$|a| \pm |b| \neq |a \pm b|$$

$$|a|^2 = a^2$$

$$\text{If } |a| < |b|, |a|^2 < |b|^2, a^2 < b^2$$

$$\text{If } |a| > |b|, |a|^2 > |b|^2, a^2 > b^2$$

Q1. Solve the following inequalities

(a)  $|2x + 1| > 7$

(b)  $|1 - x| < 3$

(c)  $|2 - x| \leq |3 + 2x|$

1(a)

$$|2x + 1| > 7$$

Using Modulus Function Property:

If  $|f(x)| > k$ , it will also imply  $f(x) < -k$  OR  $f(x) > k$

$$2x + 1 < -7 \text{ OR } 2x + 1 > 7$$

$$2x < -8 \text{ OR } 2x > 6$$

$$x < -4 \text{ OR } x > 3$$

1(b)

$$|1 - x| < 3$$

Using Modulus Function Property: If  $|f(x)| < k$ , it will also imply  $-k < f(x) < k$

$$-3 < 1 - x < 3$$

$$-4 < -x < 2$$

Multiplying negative sign to the entire inequality, we get the following:

$$4 > x > -2$$

1(c)

$$|2 - x| \leq |3 + 2x|$$

Using Modulus Function Property: If  $|a| < |b|$ ,  $|a|^2 < |b|^2$ ,  $a^2 < b^2$

$$|2 - x|^2 \leq |3 + 2x|^2$$

$$[2^2 - 2(2)(x) + x^2] \leq [3^2 + 2(3)(2)x + 4x^2]$$

$$4 - 4x + x^2 \leq 9 + 12x + 4x^2$$

$$-5 - 16x - 3x^2 \leq 0$$

$$-[(3x + 1)(x + 5)] \leq 0$$

Multiplying negative sign to the entire inequality, we get the following:

$$(3x + 1)(x + 5) \geq 0$$

$$x \geq -\frac{1}{3} \text{ OR } x \leq -5$$

Q2. Solve the following inequalities

(a)  $|x - 2| \geq x^2$

(b)  $\left| \frac{2x+1}{x-3} \right| > 1$

2(a)  $|x - 2| \geq x^2$

Using Modulus Function Property:

If  $|f(x)| > k$ , it will also imply  $f(x) < -k$  OR  $f(x) > k$

$$x - 2 \leq -x^2 \text{ OR } x - 2 \leq x^2$$

$$x - 2 + x^2 \leq 0 \text{ OR } x - 2 - x^2 \leq 0$$

$$x^2 + x - 2 \leq 0 \text{ OR } -x^2 + x - 2 \leq 0$$

$$(x + 2)(x - 1) \leq 0$$

$$-2 \leq x \leq 1$$

Multiplying negative sign to the whole inequality, we get

$$-x^2 + x - 2 \leq 0$$

$$x^2 - x + 2 \geq 0$$

Using quadratic discriminant rule, we calculate the discriminant to get the following:

$b^2 - 4ac = (-1)^2 - 4(1)(2) = 1 - 8 = -7 \therefore$  proving that this inequality has no real values that can satisfy itself.

As a result, the only range of value that can satisfy the initial inequality is:

$$-2 \leq x \leq 1$$

2(b)  $\left| \frac{2x+1}{x-3} \right| > 1$

Using Modulus Function Property,  $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$

$$\frac{|2x + 1|}{|x - 3|} > 1$$

Since the denominator is always positive as denoted by the absolute sign, we can be absolutely sure the denominator will not cause any signs to change when multiplied to both sides, we get the following as a result:

$$|2x + 1| > |x - 3|$$

Using Modulus Function Property: If  $|a| > |b|$ ,  $|a|^2 > |b|^2$ ,  $a^2 > b^2$

$$|2x + 1|^2 > |x - 3|^2 = (2x + 1)^2 > (x - 3)^2$$

$$4x^2 + 4x + 1 > x^2 - 6x + 9$$

$$3x^2 + 10x - 8 > 0$$

$$x < -4 \text{ OR } x > \frac{2}{3}$$

Going back to the original inequality, we have to exclude anything can will cause denominator values to be 0,  $\therefore x \neq 3$  and hence there have to be two more solutions that would satisfy the initial inequality,  $x < 3$  and  $x > 3$

$$\text{Therefore, } x < -4, \frac{2}{3} < x < 3, x > 3$$

Q3. Solve the following inequality

$$(a) \frac{4}{|x|} > x + 3$$

Since the numerator of the LHS is positive, we can argue that  $|4| = 4$  and put the absolute sign on numerator to get the following value:

$$\frac{|4|}{|x|} > x + 3 \text{ which can be rewritten as } \left| \frac{4}{x} \right| > x + 3$$

Using Modulus Function Property:

If  $|f(x)| > k$ , it will also imply  $f(x) < -k \text{ OR } f(x) > k$

$$\frac{4}{x} < -(x + 3) \text{ OR } \frac{4}{x} > x + 3$$

$$\frac{4}{x} < -x - 3 \text{ OR } \frac{4}{x} > x + 3$$

$$\frac{4}{x} < \frac{(-x - 3)(x)}{x} \text{ OR } \frac{4}{x} > \frac{(x + 3)(x)}{x}$$

$$\frac{4}{x} < \frac{-x^2 - 3x}{x} \text{ OR } \frac{4}{x} > \frac{x^2 + 3x}{x}$$

$$\frac{4}{x} - \frac{-x^2 - 3x}{x} < 0 \text{ OR } \frac{4}{x} - \frac{x^2 + 3x}{x} > 0$$

$$\frac{4 + x^2 + 3x}{x} < 0 \text{ OR } \frac{4 - x^2 - 3x}{x} > 0$$

$x^2 + 3x + 4$  has a discriminant value of  $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 = -7$  and  $a > 0$ . Therefore, no values could satisfy the inequality  $4 + x^2 + 3x < 0$

$$\frac{1}{x} < 0$$

$$x < 0$$

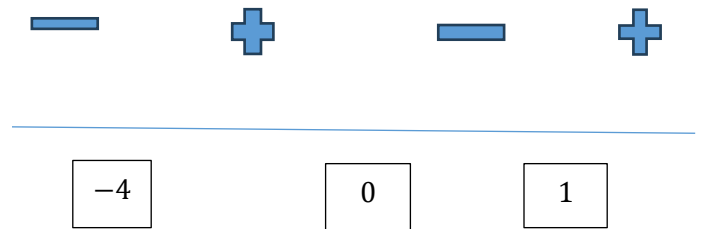
$$\frac{-x^2 - 3x + 4}{x} > 0$$

Multiplying both sides by  $-1$ , we get the following

$$\frac{x^2 + 3x - 4}{x} < 0$$

Factorizing the numerator, we get the following

$$\frac{[(x + 4)(x - 1)]}{x} < 0$$



$$x < -4 \text{ OR } 0 < x < 1$$

| Sign Test | Result             |
|-----------|--------------------|
| -5        | -1.2<br>(Negative) |
| -2        | 3<br>(Positive)    |
| 0.5       | -4.5<br>(Negative) |
| 2         | 3<br>(Positive)    |

$$\therefore x < 0 \text{ or } 0 < x < 1$$

Q4.  $\frac{|x|+5}{2-|x|} < 3$

Substitute  $u = |x|$  and we get the following:

$$\frac{u+5}{2-u} < 3$$

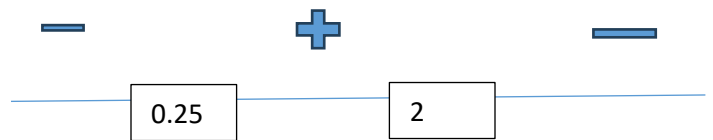
$$\frac{u+5}{2-u} - 3 < 0$$

$$\frac{u+5}{2-u} - \frac{3(2-u)}{2-u} < 0$$

$$\frac{u+5-6+3u}{2-u} < 0$$

$$\frac{4u-1}{2-u} < 0$$

$$u < \frac{1}{4} \text{ OR } u > 2$$



| Sign Test | Results          |
|-----------|------------------|
| 0         | -0.5 (Negative)  |
| 0.5       | 0.667 (Positive) |
| 3         | -11 Negative     |

For  $u = |x| < \frac{1}{4}$ , can be rewritten as  $-\frac{1}{4} < x < \frac{1}{4}$

For  $u = |x| > 2$ , can be rewritten as  $x < -2$  OR  $x > 2$

$$\therefore x < -2 \text{ OR } -\frac{1}{4} < x < \frac{1}{4} \text{ OR } x > 2$$



|             |                                                                                                                                                                                                                           |
|-------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Title       | Differentiation of Inverse Trigonometric Functions                                                                                                                                                                        |
| Author      | AprilDolphin                                                                                                                                                                                                              |
| Date        | 19/9/2024                                                                                                                                                                                                                 |
| Assumptions | This article assumes you already have good knowledge of using chain rule to differentiate functions as well as basic knowledge of finding derivatives of algebraic, logarithmic, exponential and trigonometric functions. |

| Function              | Derivatives                       |
|-----------------------|-----------------------------------|
| $f(x) = \sin^{-1}(x)$ | $f'(x) = \frac{1}{\sqrt{1-x^2}}$  |
| $f(x) = \cos^{-1}(x)$ | $f'(x) = \frac{-1}{\sqrt{1-x^2}}$ |
| $f(x) = \tan^{-1}(x)$ | $f'(x) = \frac{1}{1+x^2}$         |

| Chain Rule in General                           |
|-------------------------------------------------|
| Given $f(x) = g(h(x))$<br>$f'(x) = g'(x) h'(x)$ |

| Application of Chain Rule to Inverse Trigonometric Functions |                                        |
|--------------------------------------------------------------|----------------------------------------|
| Function                                                     | Derivatives                            |
| $f(x) = \sin^{-1}(u)$                                        | $f'(x) = \frac{1}{\sqrt{1-u^2}} (u')$  |
| $f(x) = \cos^{-1}(u)$                                        | $f'(x) = \frac{-1}{\sqrt{1-u^2}} (u')$ |
| $f(x) = \tan^{-1}(u)$                                        | $f'(x) = \frac{1}{1+u^2} (u')$         |

Example 1.

Find the derivative of the following functions

(a)  $y = \sin^{-1}(e^{6t})$

Using chain rule  $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$

$$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}} \times 6e^{6t} = \frac{6e^{6t}}{\sqrt{1-e^{12t}}}$$

(b)  $y = \tan^{-1}(2^x)$

Using chain rule  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{1}{1+(2^x)^2} \times 2^x \ln(2) = \frac{2^x \ln(2)}{1+2^{2x}}$$

(c)  $f(x) = \cos^{-1}\left(\frac{x}{3}\right)$

Using chain rule, where any composite function  $f(x) = g(h(x))$  will produce a derivative of  $g'(x)h'(x)$ .

$$\cos^{-1}\left(\frac{x}{3}\right) = \cos^{-1}\left(\frac{1}{3}x\right)$$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{3}x\right)^2}} \times \frac{1}{3} = -\frac{1}{3\sqrt{1-\frac{1}{9}x^2}} = -\frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}$$

$$= -\frac{1}{\sqrt{9-\frac{\sqrt{9^2}x^2}{9}}} = -\frac{1}{\sqrt{9-x^2}}$$

(d)  $y = e^{\sin^{-1}(2x)}$

Using chain rule, where  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$2e^{\sin^{-1}(2x)} \times \frac{1}{\sqrt{1-(2x)^2}} = \frac{2e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}$$

|        |                                       |
|--------|---------------------------------------|
| Title  | Differentiation of Implicit Functions |
| Author | AprilDolphin                          |
| Date   | 27/9/2024                             |

Definition of an explicit function, is where

$y$  can be defined as exactly equal to  $f(x)$  or written mathematically,  $y = f(x)$

Definition of an implicit function is where

$y$  cannot be written as equal to solely just  $f(x)$  but requiring the person to write or rewrite a function as where  $g(x, y) = 0$

| Examples of Explicit Functions  | Examples of Implicit Functions           |
|---------------------------------|------------------------------------------|
| $y = x^2 + \ln(x) + 2x + 4$     | $x^2 + y^2 + 2xy = 25$                   |
| $y = \sin(x^4) + x^5 - \tan(x)$ | $y^3 - \frac{5x}{y} = 6x^2y^6 + 9x - 64$ |
| $y = e^{\sin(x)} - \cos(x) + 5$ | $y^5 + \sin(xy) = x^2 + y^5$             |

Rules of Differentiation Necessary to Ensure a Smooth Understanding of Implicit Differentiation:

Chain Rule:  $f(x) = g(h(x)), f'(x) = g'(x) h'(x)$

Product Rule:  $f(x) = g(x) h(x), f'(x) = g'(x) h(x) + h'(x) g(x)$

Quotient Rule:  $f(x) = \frac{g(x)}{h(x)}, f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

Instruction to Implicit Differentiation

1. Differentiate left side and right side of implicit function with respect to  $x$
2. Identify term where  $y$  is differentiated, multiplying the derivative of  $f(y)$  by  $\frac{dy}{dx}$
3. Group terms with  $\frac{dy}{dx}$  together on one side and without on the other side
4. Take out common factor of  $\frac{dy}{dx}$  on the side with the term of  $\frac{dy}{dx}$
5. Remove terms without the factor of  $\frac{dy}{dx}$  by division, multiplication or any applicable method

Example 1.

Find the derivative of the following implicit function

$$y^2x^2 + x = y - 10$$

Step 1. Differentiate left hand side and right-hand side of implicit function with respect to  $x$ .

$$\frac{d}{dx} [y^2x^2 + x] = \frac{d}{dx} [y - 10]$$

Step 2. Differentiating term by term, we get the following and can identify the term where  $y$  is differentiated then multiply the derivative by  $\frac{dy}{dx}$

| Terms                                     | Derivatives                                  |
|-------------------------------------------|----------------------------------------------|
| $y^2x^2$ using product rule $\rightarrow$ | $[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2)$ |
| $x$ using power rule $\rightarrow$        | 1                                            |
| $y$                                       | $1 \times \frac{dy}{dx}$                     |
| $-10$                                     | 0                                            |

Putting all derivative together, we would get the following value:

$$[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2) + 1 = 1 \times \frac{dy}{dx}$$

Step 3. Group terms with  $\frac{dy}{dx}$  together on one side and without on the other side

$$2yx^2 \left( \frac{dy}{dx} \right) + 2xy^2 + 1 = \frac{dy}{dx}$$

$$2yx^2 \left( \frac{dy}{dx} \right) - \frac{dy}{dx} = -2xy^2 - 1$$

Step 4. Take out common factor of  $\frac{dy}{dx}$  on the side with the term of  $\frac{dy}{dx}$

$$(2yx^2 - 1) \left( \frac{dy}{dx} \right) = -2xy^2 - 1$$

Step 5. Remove terms without the factor of  $\frac{dy}{dx}$  by division, multiplication or any available method

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2yx^2 - 1}$$

Example 2. Differentiate the following implicit function

$$y^2 = 2x^3 + y + 7$$

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[2x^3 + y + 7]$$

$$2y \left( \frac{dy}{dx} \right) = 6x^2 + 1 \left( \frac{dy}{dx} \right) + 0$$

$$2y \left( \frac{dy}{dx} \right) - 1 \left( \frac{dy}{dx} \right) = 6x^2$$

$$\frac{dy}{dx}(2y - 1) = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{2y - 1}$$

Example 3.

$$2xy + y^2 - 3x - x^2y^5 = 0$$

$$\frac{d}{dx}[2xy + y^2 - 3x - x^2y^5] = \frac{d}{dx}[0]$$

| Terms     | Derivative                                 |
|-----------|--------------------------------------------|
| $2xy$     | $2x(1) \times \frac{dy}{dx} + 2y$          |
| $y^2$     | $2y \times \frac{dy}{dx}$                  |
| $-3x$     | $-3$                                       |
| $-x^2y^5$ | $-2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx}$ |

|   |   |
|---|---|
| 0 | 0 |
|---|---|

$$2x \left( \frac{dy}{dx} \right) + 2y + 2y \left( \frac{dy}{dx} \right) - 3 - 2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx} = 0$$

$$2x \left( \frac{dy}{dx} \right) + 2y \left( \frac{dy}{dx} \right) + 2y - 3 - 2xy^5 - 5x^2y^4 \left( \frac{dy}{dx} \right) = 0$$

$$2x \left( \frac{dy}{dx} \right) + 2y \left( \frac{dy}{dx} \right) - 5x^2y^4 \left( \frac{dy}{dx} \right) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx} (2x + 2y - 5x^2y^4) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx} = \frac{-2y + 3 + 2xy^5}{2x + 2y - 5x^2y^4}$$

Example 4.

$$y + y^2 + xy = 5 - \sin(e^{3y})$$

$$\frac{d}{dx} [y + y^2 + xy] = \frac{d}{dx} [5 - \sin(e^{3y})]$$

$$1 \left( \frac{dy}{dx} \right) + 2y \left( \frac{dy}{dx} \right) + y + x \left( \frac{dy}{dx} \right) = 0 - 3e^{3y} \cos e^{3y} \left( \frac{dy}{dx} \right)$$

$$1 \left( \frac{dy}{dx} \right) + 2y \left( \frac{dy}{dx} \right) + x \left( \frac{dy}{dx} \right) + 3e^{3y} \cos(e^{3y}) \left( \frac{dy}{dx} \right) = -y$$

$$\frac{dy}{dx} [1 + 2y + x + 3e^{3y} \cos(e^{3y})] = -y$$

$$\frac{dy}{dx} = - \frac{y}{1 + 2y + x + 3e^{3y} \cos(e^{3y})}$$

|        |                            |
|--------|----------------------------|
| Title  | Parametric Differentiation |
| Author | AprilDolphin               |
| Date   | 11/11/2024                 |

### Definition of Parametric Functions

A parametric function is a function that express relationships of 2 variables using a third variable, in which the third variable is called a parameter.

Example of a Parametric Function as follows

$$y = t^2 + 5t - 4$$

$$x = 2t + 6$$

In the above case, the relationship between  $x$  and  $y$  is expressed using parameter  $t$ .

### Simplest examples of applications of parametric functions:

Kinematics Modeling – where a flying object (such as a bird or fighter jet) trajectory creates a situation where the object's forward motion speeds up and the height follows a curve that goes up and down repeatedly.

In this above case, it would be extremely difficult to express the variables in the non-parametric way and using a parametric function for kinematic modelling is appropriate in this case in the following manner.

The forward motion (being in acceleration) over time can be written as

$$x(t) = 0.8t + 3t^2$$

The altitude of the flying object in up and down motion over time can be written as

$$y(t) = 5 + \cos(t)$$

Differentiation involving parametric functions can be done using a technique derived from chain rule.

Let  $x$  be the first variable,  $y$  be the second variable and  $t$  be the parameter involved and our goal is to find the change in  $y$  with respect to  $x$ .

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dt} \times \frac{dt}{dx}$$

1. Find  $\frac{dy}{dx}$  given that  $y = 4t^2 + 3t - 1$  and  $x = 3t^4 - 2t$

$$\frac{dy}{dt} = 8t + 3$$

$$\frac{dx}{dt} = 12t^3 - 2$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{(8t + 3)}{(12t^3 - 2)}$$

2. Find  $\frac{dy}{dx}$  for the parametric equation  $x = 2 \sin(t)$  and  $y = 3 \cos(t)$

$$\frac{dy}{dt} = -3 \sin(t)$$

$$\frac{dx}{dt} = 2 \cos(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3 \sin(t)}{2 \cos(t)} = -\frac{3}{2} \tan(t)$$



3. A curve is defined parametrically by  $x = \frac{2t}{t+1}$  and  $y = \frac{t^2}{t+1}$ .

Find  $\frac{dy}{dx}$ .

Using parametric differentiation, we can find the derivatives of  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$  as follows.

Quotient rule where  $\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\frac{dy}{dt} = \frac{(t+1)(2t) - 1(t^2)}{(t+1)^2} = \frac{2t^2 + 2t - t^2}{(t+1)^2} = \frac{t^2 + 2t}{(t+1)^2}$$

$$\frac{dx}{dt} = \frac{(t+1)(2) - 2t(1)}{(t+1)^2} = \frac{2t + 2 - 2t}{(t+1)^2} = \frac{2}{(t+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 + 2t}{(t+1)^2} \times \frac{(t+1)^2}{2} = \frac{t^2 + 2t}{2}$$

|        |                      |
|--------|----------------------|
| Title  | Integration by Parts |
| Author | AprilDolphin         |
| Date   | 26/10/2024           |

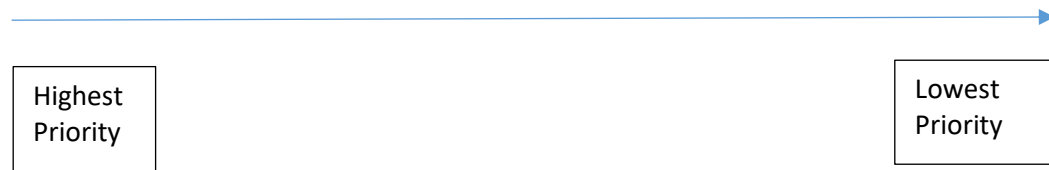
Formula for Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

In any integration by parts application, we choose the value of  $u$  using a rule where the type of function mentioned first has the highest priority and the one mentioned second has lower priority and so on, the one mentioned the last will have the lowest priority.

The rule is called

“Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, Exponential”



In short, the rule is called the L.I.A.T.E rule.

The purpose of complying to the above rule is to ensure the function we selected is made easy by integration by parts rather than being made even more complicated. As seen with the below example.

Example 1.

$$\int x e^x \, dx$$

If we select  $u = x$ , in compliance with the rule mentioned above, we literally get the following:

$$u = x \text{ and } dv = e^x$$

$$v = \int e^x = e^x$$

$$\int v = e^x$$

$$du = 1 \, dx$$

As a result, we combined them together using the integration by parts formula

$$\int u \, dv = uv - \int v \, du$$

$$x e^x - e^x(1) = x e^x - e^x + C$$

However, if we reversed the roles of  $u$  and  $dv$  above, we get the following

$$u = e^x, dv = x$$

$$du = e^x, v = \frac{x^2}{2}$$

Combining them together, using the integration by parts formula, we get the following integral, literally,

$$e^x \left( \frac{x^2}{2} \right) - \int \frac{x^2}{2} e^x \, dx$$

Which becomes as difficult or even harder to integrate compared to the function we first being with and that's not what we want.

Question 1.

Find  $\int \ln x \, dx$

Which can be rewritten as

$$\int \ln x (1) \, dx$$

Using the L.I.A.T.E rule explained earlier, we can choose  $u = \ln x$  and therefore,  $dv = 1$ .

$$u = \ln x$$

$$v = x$$

$$\int x \frac{1}{x} \, dx = \int 1 \, dx = x$$

Combining them, we get  $x \ln(x) - x + C$

Question 2.

Find  $\int x^2 \ln x \, dx$

Selecting  $u = \ln x$  and therefore  $dv = x^2$

$$v = \int x^2 = \frac{x^3}{3}$$

$$\int v \, du = \int \frac{x^3}{3} \left( \frac{1}{x} \right) = \int \frac{1}{3} x^2 = \frac{\left( \frac{1}{3} \right)}{2+1} x^{2+1} = \frac{1}{9} x^3$$

$$\ln(x) \left( \frac{x^3}{3} \right) - \frac{x^3}{9} + c$$

Question 3.

Find  $\int x \cos x \, dx$

Selecting  $u = x$  and therefore  $dv = \cos x$

$$v = \int \cos x = \sin x$$

$$\int v \, du = \int \sin(x) \, 1 \, dx = -\cos x$$

$$x \sin x + \cos x + C$$

Question 4.

Find  $\int \tan^{-1} x \, dx$

Select  $u = \tan^{-1} x$  and  $dv = 1 \, dx$

$$v = x$$

$$\int v \, du = \int x \left( \frac{1}{x^2 + 1} \right) = \int \frac{x}{x^2 + 1} = 0.5 \ln(x^2 + 1)$$

$$x \tan^{-1}(x) - 0.5 \ln(x^2 + 1)$$

|        |                                                                                                                                 |
|--------|---------------------------------------------------------------------------------------------------------------------------------|
| Title  | Differential Equations – Basic Concepts and First Order Differential Equations (Direct Integration and Separation of Variables) |
| Author | AprilDolphin                                                                                                                    |
| Date   | 30/10/2024                                                                                                                      |

Differential Equations are equations that contain derivatives, where the so-called “Solutions” are the function(s) that can satisfy the derivatives. The act of finding the indefinite integral of derivatives is generally considered solving simple differential equations.

Concepts to understand in any differential equations

- Order – the order of a differential equation is simply the order of the highest derivative in the differential equation.
- Degree – the degree of a differential equation is simply the power of the highest derivative of the differential equation.

Example

1. Given the following three examples, state the order and degree of the differential equation:

$$(a) x \left( \frac{d^3 y}{dx^3} \right) + (3y)^2 = 6x - 8$$

$$(b) 3x(y'')^2 - y^2 y' = 0$$

$$(c) x \left( \frac{dy}{dx} \right) + 8 \left( \frac{d^2 y}{dx^2} \right)^3 = 3x - 5y$$

1(a) Highest derivative is the third derivative – third order differential equation  
Highest power of the highest derivative is 1 – first degree differential equation  
∴ Third order, first degree differential equation.

1(b) Highest derivative is the second derivative – second order differential equation  
Highest power of the highest derivative is 2 – second degree differential equation  
∴ Second order, second degree differential equation

1(c) Highest derivative is the second derivative – second order differential equation  
Highest power of the highest derivative is 3 – third degree differential equation  
∴ Second order, third degree differential equation.

After understanding the above, we can go on and understand the concept of independent and dependent variables.

Independent variables in any differential equation are the variables with respect to where differentiation occur and dependent variables are the variables being differentiated.

In the below example, identify the dependent and independent variables.

Example

2.

$$x \left( \frac{d^3 y}{dx^3} \right) + (3y)^2 = 6x - 8$$

$y$  is the dependent variable

$x$  is the independent variable

Q1. Solve the following differential equation using direct integration

(a)  $\frac{dy}{dx} = \ln x$

Move  $dx$  to the other side by multiplying both sides by  $dx$  to get the following  
 $dy = \ln x \, dx$

Integrate both sides to get

$$\int dy = \int \ln x \, dx$$

Using integration by parts for the right-hand side, we get the following

Set  $u = \ln x$  ,  $dv = 1$

$$v = x$$

$$du = \frac{1}{x}$$

$$\int v \, du = \int \frac{x}{x} = \int 1 = x$$

$$y = x(\ln x) - x + C$$

Q2. Solve the following differential equation by separation of variables.

$$x + y \left( \frac{dy}{dx} \right) = 2$$

$$y \left( \frac{dy}{dx} \right) = 2 - x$$

Multiply both sides by  $dx$

$$y \, dy = 2 - x \, dx$$

Integrate both sides to get the following

$$\int y \, dy = \int 2 - x \, dx$$

$$\frac{y^2}{2} = 2x - \frac{x^2}{2}$$

$$y^2 = 4x - x^2 + C$$

Q3. Solve the following differential equation by separation of variables.

$$\frac{dy}{dx} = \frac{x^2}{y^3}$$

Cross-multiply both sides denominator and we get the following

$$y^3 \, dy = x^2 \, dx$$

$$\int y^3 \, dy = \int x^2 \, dx$$

$$\frac{y^4}{4} = \frac{x^3}{3}$$

Cross multiply again to get

$$3y^4 = 4x^3$$

Divide both sides by 3

$$y^4 = \frac{4}{3}x^3 + C$$